

# Generalized Fourier transform on Chébli-Trimèche hypergroups

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**Abstract** In this paper, we prove the Hardy-Littlewood inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

**Keywords** Chébli-Trimèche hypergroups · Generalized Fourier transform · Jacobi hypergroup · Jacobi function

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## 1 Introduction

We consider the Chébli-Trimèche hypergroup  $(\mathbb{R}_+, *(A))$  associated with the function  $A$  which depends on a real parameter  $\alpha > -\frac{1}{2}$  (see next section). We prove the Hardy-Littlewood inequality for the generalized Fourier transform  $\mathcal{F}(f)$  of a function  $f$  in  $L^p(\mathbb{R}_+, A(x)dx)$ ,  $1 < p \leq 2$ . Next, inspired by the definition of usual Besov spaces and Besov-Dunkl spaces (see [2, 5]), we define the Besov-type spaces for Chébli-Trimèche hypergroup denoted by  $\mathcal{B}_{\gamma, \alpha}^{p, q}$ , as the subspace of functions  $f \in L^p(\mathbb{R}_+, A(x)dx)$  satisfying

$$\int_0^{+\infty} \left( \frac{\omega_{A,p}(f, x)}{x^\gamma} \right)^q \frac{dx}{x} < +\infty \quad \text{if } q < +\infty$$

and

$$\sup_{x \in ]0, +\infty[} \frac{\omega_{A,p}(f, x)}{x^\gamma} < +\infty \quad \text{if } q = +\infty,$$

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where  $\omega_{A,p}(f, x) = \|\tau_x(f) - f\|_{A,p}$  is the modulus of continuity of first order of  $f$  with  $\tau_x$  the generalized translation operators,  $x \in \mathbb{R}_+$  (see next section). We establish in the particular case of Jacobi hypergroup further results concerning integrability of the generalized Fourier transform  $\mathcal{F}(f)$  of a function  $f$  when  $f$  belongs to a suitable Besov-type spaces. Analogous results have been obtained for the theory of Dunkl operators in [1, 3, 4].

The contents of this paper are as follows.

In section 2, we collect some results about harmonic analysis on Chébli-Trimèche hypergroups.

In section 3, we prove the Hardy-Littlewood inequality for the generalized Fourier transform on Chébli-Trimèche hypergroups and we study in the particular case of the Jacobi hypergroup the integrability of this transform on Besov-type spaces.

Along this paper we use  $c$  to denote a suitable positive constant which is not necessarily the same in each occurrence. Furthermore, we denote by

- $\mathbb{C}_{*,c}(\mathbb{R})$  the space of even continuous functions on  $\mathbb{R}$ , with compact support.
- $\mathcal{D}_*(\mathbb{R})$  the space of even  $C^\infty$ -functions on  $\mathbb{R}$  with compact support.

## 2 Preliminaries

In this section, we recall some notations and results about harmonic analysis on Chébli-Trimèche hypergroups and we refer for more details to the articles [6, 9, 11, 12].

Let  $A$  be the Chébli-Trimèche function defined on  $\mathbb{R}_+$  and satisfying the following conditions.

- i)  $A(x) = x^{2\alpha+1}B(x)$ , with  $\alpha > -\frac{1}{2}$ , and  $B$  an even  $C^\infty$ -function on  $\mathbb{R}$  such that  $B(x) \geq 1$  for all  $x \in \mathbb{R}_+$ .
- ii)  $A$  is increasing and unbounded.
- iii)  $\frac{A'}{A}$  is decreasing on  $\mathbb{R}_+^* = ]0, +\infty[$  and  $\lim_{x \rightarrow +\infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$ .
- iv) There exists a constant  $\eta > 0$  such that for all  $x \in [x_0, +\infty[$ ,  $x_0 > 0$ , we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\eta x}F(x) & , \text{ if } \rho > 0 \\ \frac{2\alpha+1}{x} + e^{-\eta x}F(x) & , \text{ if } \rho = 0, \end{cases}$$

where  $F$  is a  $C^\infty$ -function bounded together with its derivatives.

We consider the Chébli-Trimèche hypergroup  $(\mathbb{R}_+, *(A))$  associated with the function  $A$ . We note that it is commutative with neutral element 0 and the identity mapping is the involution. The Haar measure  $m$  on  $(\mathbb{R}_+, *(A))$  is absolutely continuous with respect to the Lebesgue measure and can be chosen to have the Lebesgue density  $A$ .

*Remark 1* If  $A(x) = 2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}$ , with  $\alpha \geq \beta \geq -\frac{1}{2}$ ,  $\alpha \neq -\frac{1}{2}$  and  $\rho = \alpha + \beta + 1$ ,  $(\mathbb{R}_+, *(A))$  is called the Jacobi hypergroup.

Let  $\Delta$  be the differential operator on  $\mathbb{R}_+^*$  given by

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

The solution  $\varphi_\lambda, \lambda \in \mathbb{C}$ , of the differential equation

$$\begin{cases} \Delta u(x) = -(\lambda^2 + \rho^2)u(x), \\ u(0) = 1, \frac{d}{dx}u(0) = 0, \end{cases}$$

is multiplicative on  $(\mathbb{R}_+, *(A))$  in the sense that

$$\forall x, y \in \mathbb{R}_+, \int_{\mathbb{R}_+} \varphi_\lambda(t) d(\delta_x * \delta_y)(t) = \varphi_\lambda(x) \varphi_\lambda(y),$$

where  $\delta_x$  is the point mass at  $x$  and  $\delta_x * \delta_y$  is a probability measure which is absolutely continuous with respect to the measure  $m$  and satisfies

$$\text{supp } \delta_x * \delta_y = [|x - y|, x + y].$$

We list some known properties of the characters  $\varphi_\lambda$  of the hypergroups.

- i) For each  $\lambda \in \mathbb{C}$ , the function  $x \mapsto \varphi_\lambda(x)$  is an even  $C^\infty$ -function on  $\mathbb{R}$  and for each  $x \in \mathbb{R}_+$ , the function  $\lambda \mapsto \varphi_\lambda(x)$  is an entire function on  $\mathbb{C}$ .
- ii) For every  $\lambda \in \mathbb{C}$ , the function  $\varphi_\lambda$  admits the integral representation

$$\forall x \in \mathbb{R}_+^*, \quad \varphi_\lambda(x) = \int_0^x K(x, y) \cos(\lambda y) dy.$$

Where  $K(x, \cdot)$  is a positive even  $C^\infty$ -function on  $] -x, x[$  with support in  $[-x, x]$ .

*Remark 2* In the Jacobi hypergroup (see Remark 1), we have for all  $x \in \mathbb{R}_+$  and  $\lambda \in \mathbb{C}$ ,

$$\varphi_\lambda(x) = \varphi_\lambda^{(\alpha, \beta)}(x) = {}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1; -\sinh^2 x\right),$$

where  ${}_2F_1$  is the Gauss hypergeometric function (see [9]). The function  $\varphi_\lambda^{(\alpha, \beta)}(x)$  is the Jacobi function and it satisfies for all  $\lambda \in \mathbb{R}$  and  $t \in \mathbb{R}_+^*$

$$|1 - \varphi_\lambda^{(\alpha, \beta)}(t)| \geq c \min\{1, (\lambda t)^2\}, \quad (1)$$

where  $c$  is constant which depends only on  $\alpha$  and  $\beta$  (see [7, 8]).

For every  $p \in [1, +\infty]$ , we denote by  $L_A^p(\mathbb{R}_+)$  the space  $L^p(\mathbb{R}_+, A(x)dx)$  and by  $L_c^p(\mathbb{R}_+)$  the space  $L^p(\mathbb{R}_+, \frac{d\lambda}{|c(\lambda)|^2})$  where  $|c(\lambda)|^{-2}$  is an even continuous function on  $\mathbb{R}$ , satisfying the estimates: There exist positive constants  $k, k_1, k_2$  such that

- i) If  $\rho = 0$  and  $\alpha > 0$  then

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{C}. \quad (2)$$

- ii) If  $\rho > 0$  and  $\alpha > -\frac{1}{2}$  then

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{C}, |\lambda| > k, \quad (3)$$

and

$$k_1 |\lambda|^2 \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^2, \quad \lambda \in \mathbb{C}, |\lambda| \leq k. \quad (4)$$

We use  $\|\cdot\|_{A,p}$  and  $\|\cdot\|_{c,p}$  as a shorthand respectively of  $\|\cdot\|_{L_A^p(\mathbb{R}_+)}$  and  $\|\cdot\|_{L_c^p(\mathbb{R}_+)}$ .

For  $f \in L_A^1(\mathbb{R}_+)$  the generalized Fourier transform of  $f$  is given by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x) \varphi_\lambda(x) A(x) dx.$$

The generalized Fourier transform satisfies the following properties.

i) For  $f \in L_A^1(\mathbb{R}_+)$ , we have

$$\|\mathcal{F}(f)\|_{c,\infty} \leq \|f\|_{A,1} \quad (5)$$

ii) For  $f$  in  $L_A^1(\mathbb{R}_+)$  such that  $\mathcal{F}(f)$  belongs to  $L_c^1(\mathbb{R}_+)$ , we have the following inversion formula for the transform  $\mathcal{F}$

$$f(x) = \int_{\mathbb{R}_+} \mathcal{F}(f)(\lambda) \phi_\lambda(x) \frac{d\lambda}{|c(\lambda)|^2}, \text{ a.e.}$$

iii) (Plancherel formula) For all  $f \in \mathcal{D}_*(\mathbb{R})$ , we have

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x) dx = \int_{\mathbb{R}_+} |\mathcal{F}(\lambda)|^2 \frac{d\lambda}{|c(\lambda)|^2}. \quad (6)$$

The transform  $\mathcal{F}$  can be uniquely extended to an isometric isomorphism from  $L_A^2(\mathbb{R}_+)$  onto  $L_c^2(\mathbb{R}_+)$ .

For  $1 \leq p \leq 2$ , we denote by  $p'$  the conjugate of  $p$ . From (2.5), (2.6) and the Marcinkiewicz interpolation theorem (see [10]), we obtain for  $f \in L_A^p(\mathbb{R}_+)$

$$\mathcal{F}(f) \in L_c^{p'}(\mathbb{R}_+). \quad (7)$$

For  $x \in \mathbb{R}_+$  and  $f \in \mathbb{C}_{*,c}(\mathbb{R})$ , the generalized  $x$ -translate of  $f$  is defined by

$$\forall y \in \mathbb{R}_+, \quad \tau_x f(y) = \int_{\mathbb{R}_+} f(t) d(\delta_x * \delta_y)(t),$$

and we have  $\tau_x f(0) = f(x)$ .

The generalized translation operators  $\tau_x$ ,  $x \in \mathbb{R}_+$ , satisfy the following properties.

i) For all  $x, y \in \mathbb{R}_+$  and  $\lambda \in \mathbb{C}$ , we have the product formula

$$\tau_x \phi_\lambda(y) = \phi_\lambda(x) \phi_\lambda(y).$$

ii) For  $f \in \mathcal{D}_*(\mathbb{R})$  and  $x \in \mathbb{R}_+$ , the function  $y \mapsto \tau_x f(y)$  belongs to  $\mathcal{D}_*(\mathbb{R})$  and we have

$$\forall \lambda \in \mathbb{R}_+, \quad \mathcal{F}(\tau_x f)(\lambda) = \phi_\lambda(x) \mathcal{F}f(\lambda). \quad (8)$$

iii) Let  $f \in L_A^p(\mathbb{R}_+)$ ,  $p \in [1, +\infty]$ . For all  $x \in \mathbb{R}_+$ , the function  $\tau_x f$  belongs to  $L_A^p(\mathbb{R}_+)$ ,  $p \in [1, +\infty]$ , and we have

$$\|\tau_x f\|_{A,p} \leq \|f\|_{A,p}.$$

### 3 Generalized Fourier transform

Throughout this section,  $k$  refers to the constant obtained in (3) and (4) from the estimates of  $|c(\lambda)|^{-2}$ .

In the following lemma, we prove the Hardy-Littlewood inequality for the Fourier transform.

**Lemma 1** For  $f \in L_A^p(\mathbb{R}_+)$ ,  $1 < p \leq 2$ , one has

$$\int_{\mathbb{R}_+} (g(x))^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \leq c \|f\|_{A,p}^p \quad (9)$$

where

- i)  $g(x) = x^{2(\alpha+1)}$  if  $\rho = 0$  and  $\alpha > 0$ .
  - ii)  $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leq k. \end{cases}$  if  $\rho > 0$  and  $\alpha > -\frac{1}{2}$
- where  $k$  refers to the constant obtained from the estimates of  $|c(x)|^{-2}$ .

*Proof* For  $f \in L_A^p(\mathbb{R}_+)$ ,  $1 \leq p \leq 2$ , we consider the operator

$$L(f)(x) = g(x) \mathcal{F}(f)(x), \quad x \in \mathbb{R}_+.$$

For every  $f \in L_A^2(\mathbb{R}_+)$ , we have from (6)

$$\left( \int_{\mathbb{R}_+} |L(f)(x)|^2 \frac{dx}{(g(x))^2 |c(x)|^2} \right)^{\frac{1}{2}} = \|\mathcal{F}(f)\|_{c,2} = \|f\|_{A,2},$$

hence  $L$  is an operator of strong-type  $(2, 2)$  between the spaces  $(\mathbb{R}_+, A(x)dx)$  and  $(\mathbb{R}_+, \frac{dx}{(g(x))^2 |c(x)|^2})$ .

i) Assume  $\rho = 0$ ,  $\alpha > 0$  and  $g(x) = x^{2(\alpha+1)}$ . For  $\lambda \in ]0, +\infty[$ ,  $f \in L_A^1(\mathbb{R}_+)$  and using (2) and (5), we can write

$$\begin{aligned} \int_{\{x \in \mathbb{R}_+ : |L(f)(x)| > \lambda\}} \frac{dx}{(g(x))^2 |c(x)|^2} &= \int_{\{x \in \mathbb{R}_+ : |L(f)(x)| > \lambda\}} \frac{dx}{x^{4(\alpha+1)} |c(x)|^2} \\ &\leq c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} \frac{x^{2\alpha+1}}{x^{4(\alpha+1)}} dx \\ &\leq c \frac{\|f\|_{A,1}}{\lambda}. \end{aligned}$$

It yields that  $L$  is of weak-type  $(1, 1)$  between the spaces under consideration.

By the Marcinkiewicz interpolation theorem (see [10]), we can assert that  $L$  is an operator of strong-type  $(p, p)$ ,  $1 < p \leq 2$  between the spaces  $(\mathbb{R}_+, A(x)dx)$  and  $(\mathbb{R}_+, \frac{dx}{(g(x))^2 |c(x)|^2})$ .

We conclude that,

$$\begin{aligned} \int_{\mathbb{R}_+} |L(f)(x)|^p \frac{dx}{(g(x))^2 |c(x)|^2} &= \int_{\mathbb{R}_+} |g(x)|^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \\ &\leq c \|f\|_{A,p}^p, \end{aligned}$$

which prove the result.

- ii) Suppose now  $\rho > 0$ ,  $\alpha > -\frac{1}{2}$  and  $g(x) = \begin{cases} x^{2(\alpha+1)} & \text{for } x > k \\ x^3 & \text{for } x \leq k, \end{cases}$

where  $k$  is the constant obtained in (3) and (4) from the estimates of  $|c(\lambda)|^{-2}$ . Let  $\lambda \in ]0, +\infty[$  and  $f \in L_A^1(\mathbb{R}_+)$ , by (3), (4) and (5), we have

$$\begin{aligned}
& \int_{\{x \in \mathbb{R}_+ : |L(f)(x)| > \lambda\}} \frac{dx}{(g(x))^2 |c(x)|^2} \leq \int_{\{x \in \mathbb{R}_+ : g(x) > \frac{\lambda}{\|f\|_{A,1}}\}} \frac{dx}{(g(x))^2 |c(x)|^2} \\
& \leq \int_{\{x \in \mathbb{R}_+ : g(x) > \frac{\lambda}{\|f\|_{A,1}}\}} \chi_{[0,k]}(x) \frac{dx}{(g(x))^2 |c(x)|^2} \\
& \quad + \int_{\{x \in \mathbb{R}_+ : g(x) > \frac{\lambda}{\|f\|_{A,1}}\}} \chi_{[k,+\infty[}(x) \frac{dx}{(g(x))^2 |c(x)|^2} \\
& \leq c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{3}}}^{+\infty} \chi_{[0,k]}(x) \frac{x^2}{x^6} dx + c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} \chi_{[k,+\infty[}(x) \frac{x^{2\alpha+1}}{x^{4(\alpha+1)}} dx \\
& \leq c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{3}}}^{+\infty} x^{-4} dx + c \int_{(\frac{\lambda}{\|f\|_{A,1}})^{\frac{1}{2(\alpha+1)}}}^{+\infty} x^{-2\alpha-3} dx \leq c \frac{\|f\|_{A,1}}{\lambda}.
\end{aligned}$$

Hence  $L$  is of weak-type  $(1, 1)$  between the spaces  $(\mathbb{R}_+, A(x)dx)$  and  $(\mathbb{R}_+, \frac{dx}{(g(x))^2 |c(x)|^2})$ .

We conclude by the Marcinkiewicz interpolation theorem that  $L$  is of strong-type  $(p, p)$ , between the spaces under consideration.

It yields, that

$$\begin{aligned}
\int_{\mathbb{R}_+} |L(f)(x)|^p \frac{dx}{(g(x))^2 |c(x)|^2} &= \int_{\mathbb{R}_+} |g(x)|^{p-2} |\mathcal{F}(f)(x)|^p \frac{dx}{|c(x)|^2} \\
&\leq c \|f\|_{A,p}^p,
\end{aligned}$$

thus we obtain the result.

In the following, we study the integrability of the generalized Fourier transform in the Jacobi hypergroup case (see Remarks 1 and 2). For  $1 \leq p \leq 2$ , we denote by  $p'$  the conjugate of  $p$ .

**Lemma 2** *Let  $1 \leq p \leq 2$  and  $f \in L_A^p(\mathbb{R}_+)$ . Then there exists a positive constant  $c$  such that for  $\delta > 0$ , one has*

$$\left( \int_0^{+\infty} \min\{1, (\delta x)^{2p'}\} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}} \leq c \omega_{A,p}(f)(\delta), \text{ if } 1 < p \leq 2$$

and

$$\operatorname{ess\,sup}_{x>0} \left( \min\{1, (\delta x)^2\} |\mathcal{F}(f)(x)| \right) \leq c \omega_{A,1}(f)(\delta), \text{ if } p = 1.$$

*Proof* For  $f \in L_A^p(\mathbb{R}_+)$ ,  $1 \leq p \leq 2$ , we have by (8)

$$\mathcal{F}(\tau_\delta(f) - f)(x) = (\varphi_x(\delta) - 1) \mathcal{F}(f)(x),$$

for  $\delta > 0$  and a.e  $x \in \mathbb{R}_+$ . Applying (7), we get

$$\begin{aligned}
\|\mathcal{F}(\tau_\delta(f) - f)\|_{c,p'} &= \left( \int_0^{+\infty} |1 - \varphi_x(\delta)|^{p'} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}} \\
&\leq c \omega_{A,p}(f)(\delta).
\end{aligned}$$

From (1), we obtain our results. Here when  $p = 1$ , we make the usual modification.

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*Remark 3*

- i) In the lemma 2, the gauge on the size of the generalized transform in terms of an integral modulus of continuity of  $f$  gives a quantitative form of the Riemann-Lebesgue lemma:

$$\left( \int_{\frac{1}{\delta}}^{+\infty} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}} \leq c \omega_{A,p}(f)(\delta), \text{ if } 1 < p \leq 2$$

and

$$\operatorname{ess\,sup}_{x > \frac{1}{\delta}} |\mathcal{F}(f)(x)| \leq c \omega_{A,1}(f)(\delta), \text{ if } p = 1.$$

- ii) We will use the following estimates deduced from lemma 2 to establish the integrability of  $\mathcal{F}(f)$  when  $f$  belongs in  $\mathcal{B}_{\gamma,\alpha}^{p,\infty}$  for  $1 \leq p \leq 2$ :

$$\delta^2 \left( \int_0^{\frac{1}{\delta}} x^{2p'} |\mathcal{F}(f)(x)|^{p'} \frac{dx}{|c(x)|^2} \right)^{\frac{1}{p'}} \leq c \omega_{A,p}(f)(\delta), \text{ if } 1 < p \leq 2 \quad (10)$$

and

$$\operatorname{ess\,sup}_{0 < x < \frac{1}{\delta}} \left( (\delta x)^2 |\mathcal{F}(f)(x)| \right) \leq c \omega_{A,1}(f)(\delta), \text{ if } p = 1. \quad (11)$$

**Theorem 1** *If  $f \in \mathcal{B}_{\frac{2(\alpha+1)}{p},\alpha}^{p,1} \cap \mathcal{B}_{\frac{3}{p},\alpha}^{p,1}$  for  $1 < p \leq 2$ , then*

$$\mathcal{F}(f) \in L_c^1(\mathbb{R}_+).$$

*Proof* For  $f \in L_A^p(\mathbb{R}_+)$ ,  $1 < p \leq 2$  and  $\delta > 0$ , we can write from (8) and (9)

$$\int_{\mathbb{R}_+} |1 - \varphi_t(\delta)|^p |\mathcal{F}(\tau_\delta(f)(t))|^p (g(t))^{p-2} \frac{dt}{|c(t)|^2} \leq c (\omega_{A,p}(f)(\delta))^p,$$

then by (1), we obtain

$$\delta^{2p} \int_0^{\frac{1}{\delta}} t^{2p} |\mathcal{F}(f)(t)|^p (g(t))^{p-2} \frac{dt}{|c(t)|^2} \leq c (\omega_{A,p}(f)(\delta))^p. \quad (12)$$

From (3) and (4), we have

$$\begin{aligned} & \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \frac{dt}{|c(t)|^2} \\ &= \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \chi_{[0,k]}(t) \frac{dt}{|c(t)|^2} + \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \chi_{[k,+\infty]}(t) \frac{dt}{|c(t)|^2} \\ &\leq c \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \chi_{[0,k]}(t) [t^2 dt] + c \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \chi_{[k,+\infty]}(t) [t^{2\alpha+1} dt], \end{aligned}$$

by Hölder's inequality and (12), we have

$$\begin{aligned}
& \int_0^{\frac{1}{\delta}} t |\mathcal{F}(f)(t)| \frac{dt}{|c(t)|^2} \\
& \leq c \left( \int_0^{\frac{1}{\delta}} t^{3(p-2)+2p} |\mathcal{F}(f)(t)|^p \chi_{[0,k]}(t) [t^2 dt] \right)^{\frac{1}{p}} \left( \int_0^{\frac{1}{\delta}} t^{2(p'-2)} \chi_{[0,k]}(t) dt \right)^{\frac{1}{p'}} \\
& + c \left( \int_0^{\frac{1}{\delta}} t^{2(\alpha+1)(p-2)+2p} |\mathcal{F}(f)(t)|^p \chi_{[k,+\infty]}(t) [t^{2\alpha+1} dt] \right)^{\frac{1}{p}} \\
& \times \left( \int_0^{\frac{1}{\delta}} t^{(2\alpha+1)(p'-2)+2\alpha-1} \chi_{[k,+\infty]}(t) dt \right)^{\frac{1}{p'}} \\
& \leq c \left( \int_0^{\frac{1}{\delta}} t^{2p} |\mathcal{F}(f)(t)|^p (g(t))^{p-2} \frac{dt}{|c(t)|^2} \right)^{\frac{1}{p}} \\
& \times \left\{ \left( \int_0^{\frac{1}{\delta}} t^{2(p'-2)} dt \right)^{\frac{1}{p'}} + \left( \int_0^{\frac{1}{\delta}} t^{(2\alpha+1)(p'-2)+2\alpha-1} dt \right)^{\frac{1}{p'}} \right\} \\
& \leq c \delta^{-2} \omega_{A,p}(f)(\delta) \left( \frac{1}{\delta^{\frac{3}{p}-1}} + \frac{1}{\delta^{\frac{2(\alpha+1)}{p}-1}} \right) \leq c \left( \frac{\omega_{A,p}(f)(\delta)}{\delta^{\frac{3}{p}}} \frac{1}{\delta} + \frac{\omega_{A,p}(f)(\delta)}{\delta^{\frac{2(\alpha+1)}{p}}} \frac{1}{\delta} \right).
\end{aligned}$$

Integrating with respect to  $\delta$  over  $\mathbb{R}_+$  for  $f \in \mathcal{B}_{\frac{2(\alpha+1)}{p}, \alpha}^{p,1} \cap \mathcal{B}_{\frac{3}{p}, \alpha}^{p,1}$ , the double integral is evaluated by interchanging the orders of integration, it yields

$$\int_0^{+\infty} |\mathcal{F}(f)(t)| \frac{dt}{|c(t)|^2} < +\infty.$$

This complete the proof.

**Theorem 2** Let  $\gamma > 0$ ,  $1 \leq p \leq 2$  and  $f \in \mathcal{B}_{\gamma, \alpha}^{p, \infty}$ , then

i) For  $p \neq 1$  and  $0 < \gamma \leq \frac{2(\alpha+1)}{p}$ , one has

$$\mathcal{F}(f) \in L_c^s(\mathbb{R}_+) \text{ provided that } \frac{2(\alpha+1)p}{\gamma p + 2(\alpha+1)(p-1)} < s \leq p'.$$

ii) For  $p \neq 1$  and  $\gamma > \frac{2(\alpha+1)}{p}$ , one has

$$\mathcal{F}(f) \in L_c^1(\mathbb{R}_+).$$

iii) For  $p = 1$  and  $\gamma > \sup(3, 2(\alpha+1))$ , one has

$$\mathcal{F}(f) \in L_c^1(\mathbb{R}_+).$$

*Proof* Let  $f \in \mathcal{B}_{\gamma, \alpha}^{p, \infty}$ ,  $1 \leq p \leq 2$ .

i) Suppose that  $p \neq 1$  and  $0 < \gamma \leq \frac{2(\alpha+1)}{p}$ . Let  $\frac{2(\alpha+1)p}{\gamma p + 2(\alpha+1)(p-1)} < s \leq p'$ , we define the function

$$g(t) = \int_k^t |\mathcal{F}(f)(x)|^s x^s \frac{dx}{|c(x)|^2}, \quad t > k.$$

By Hölder's inequality, (4) and (10) we have

$$\begin{aligned}
g(t) & \leq \left( \int_k^t |\mathcal{F}(f)(x)|^{p'} x^{2p'} \frac{dx}{|c(x)|^2} \right)^{\frac{s}{p'}} \left( \int_k^t \frac{dx}{|c(x)|^2} \right)^{1-\frac{s}{p'}} \\
& \leq c t^{2s} (\omega_{A,p}(f)(\frac{1}{t}))^s \left( \int_k^t \frac{dx}{|c(x)|^2} \right)^{1-\frac{s}{p'}} \\
& \leq c t^{(2-\gamma)s} \left( \int_k^t x^{2\alpha+1} dx \right)^{1-\frac{s}{p'}} \leq c t^{(2-\gamma)s+2(\alpha+1)(1-\frac{s}{p'})}.
\end{aligned}$$



Then we get

$$\begin{aligned}
\int_k^t |\mathcal{F}(f)(x)|^s \frac{dx}{|c(x)|^2} &= \int_k^t x^{-2s} g'(x) dx \\
&= t^{-2s} g(t) + 2s \int_k^t x^{-2s-1} g(x) dx \\
&\leq c \left( t^{-\gamma s + 2(\alpha+1)(1-\frac{s}{p})} + \int_k^t x^{-\gamma s + 2(\alpha+1)(1-\frac{s}{p})-1} dx \right) \\
&\leq c \left( t^{-\gamma s + 2(\alpha+1)(1-\frac{s}{p})} + 1 \right),
\end{aligned}$$

it yields that  $\mathcal{F}(f) \in L_c^s([k, +\infty[, \frac{dx}{|c(x)|^2})$ . Since  $\mathcal{F}(f) \in L^{p'}([0, k], \frac{dx}{|c(x)|^2}) \subset L^s([0, k], \frac{dx}{|c(x)|^2})$ , we deduce that  $\mathcal{F}(f)$  is in  $L_c^s(\mathbb{R}_+)$ .

ii) Assume now  $\gamma > \frac{2(\alpha+1)}{p}$ . For  $p \neq 1$ , by proceeding in the same manner as the proof of i) with  $s = 1$ , we obtain the desired result.

iii) For  $p = 1$  and  $\gamma > \sup(3, 2(\alpha+1))$ . By Hölder's inequality, (3), (4) and (11), we have for  $t > 0$

$$\begin{aligned}
\int_0^{\frac{1}{t}} |\mathcal{F}(f)(x)| x \frac{dx}{|c(x)|^2} &\leq \text{ess sup}_{0 < x \leq \frac{1}{t}} x^2 |\mathcal{F}_k(f)(x)| \int_0^{\frac{1}{t}} \frac{1}{x} \frac{dx}{|c(x)|^2} \\
&\leq c t^{\gamma-2} \left( \int_0^{\frac{1}{t}} \frac{1}{x} \chi_{\{0 \leq x \leq k\}} \frac{dx}{|c(x)|^2} + \int_0^{\frac{1}{t}} \frac{1}{x} \chi_{\{x > k\}} \frac{dx}{|c(x)|^2} \right) \\
&\leq c t^{\gamma-2} [t^{-2} + t^{-(2\alpha+1)}] \leq c [t^{(\gamma-3)-1} + t^{\gamma-2(\alpha+1)-1}].
\end{aligned}$$

Integration with respect to  $t$  over  $(0, 1)$  and applying Fubini's theorem we obtain

$$\int_1^{+\infty} |\mathcal{F}(f)(x)| \frac{dx}{|c(x)|^2} \leq c \left( \int_0^1 t^{(\gamma-3)-1} dt + \int_0^1 t^{\gamma-2(\alpha+1)-1} dt \right) < \infty.$$

Since  $L^\infty([0, 1], \frac{dx}{|c(x)|^2}) \subset L^1([0, 1], \frac{dx}{|c(x)|^2})$ , then  $\mathcal{F}(f) \in L_c^1(\mathbb{R}_+)$ .

*Remark 4* For  $\gamma > \sup(3, 2(\alpha+1))$ , we can assert from the theorem 2, iii) that  $\mathcal{B}_{\gamma, \alpha}^{1, \infty}$  is an example of space where we can apply the inversion formula.

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